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Variable Populations and the Measurement of Poverty and Inequality

A Selective Overview

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Abstract

The present paper is a selective overview, very considerably based on work in which the author himself has been involved, of the difficulties which can arise in the measurement of poverty and inequality when one compares populations of differing size. The paper begins with certain problems attending the measurement of poverty when the overall population size is fixed but the numbers of the poor are permitted to vary: one discovers a certain commonality of outcomes between Derek Parfit's quest for a satisfactory theory of wellbeing and the economist's quest for a satisfactory measure of poverty. Complications arising from both the poverty and inequality rankings of distributions when the aggregate size of the population is allowed to vary are also investigated. It is suggested in the paper that, from the perspectives of both logical consistency and ethical appeal, there are problems involved in variable population comparisons of poverty and inequality which deserve to be taken note of and enquired into.

Keywords: poverty, inequality, total principle, average principle, fixed population axioms, variable population axioms, impossibility theorems

JEL classification: I30, I32, J19, O15

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1 Introduction

There are a number of inter-related issues revolving around the measurement of poverty and inequality which arise from a consideration of certain problems addressed by the philosopher Derek Parfit in the field of population ethics (see, in particular, Parfit 1984). This article is an extended essay which presents a connected treatment of the themes just mentioned. What is attempted is an overview of the subject, but one which is selectively biased toward earlier work in which the present writer (either individually or in collaboration) has himself been involved. The paper on offer unsurprisingly draws heavily—and often enough quite directly—on the author's own work, notably Subramanian (2000, 2002b, 2005a, 2005b, 2006, 2010, 2011a, 2011b), and Hassoun and Subramanian (2011). Of related interest are the essays by, among others, Kundu and Smith (1983), Bossert (1990), Paxton (2003), Chakravarty, Kanbur and Mukherjee (2006), Kanbur and Mukherjee (2007), and Hassoun (2010). It is hoped that the paper will justify the view that there are certain distinctive and non-trivial problems in the measurement of poverty and inequality which deserve special attention when we are dealing with comparisons of distributions across variable populations, and that it would be of use to have a self-contained, if selective, summary of these issues all in one place. This is the basic motivation underlying the paper.

More specifically, this essay will have three principal parts to it. The first part will deal with the problem of poverty measurement when the overall population size is fixed but the population of the poor is allowed to vary. One discovers a striking set of analogies between Parfit's quest for a satisfactory 'theory of beneficence'—he called it Theory X—and economists' quest for a satisfactory measure of poverty. Once the analytical links between the two enterprises are established, it becomes relatively easy to see that some of Parfit's celebrated results in population ethics, such as his Repugnant Conclusion, and his critiques of the 'total' and 'average principles' in utilitarianism, can be replicated within the domain of poverty measurement. This leads to the (arguable) inference that there is a commonality of failure shared by Parfit's search for a reasonable Theory X and the search for a reasonable real-valued representation of poverty.

The second part of the essay will deal with the problems posed by poverty comparisons across variable populations: here it is not just the poor population but the entire population that is allowed to vary in size. Axioms for poverty measurement are typically laid down for fixed populations, and the bridge between fixed and variable populations is invariably established through the postulation of the so-called 'replication invariance axiom', which is widely believed to be a perfectly routine, straightforward, and innocuous restriction. However, closer scrutiny suggests that replication invariance is not as ethically unexceptionable as it may appear to be. In particular, if additions to the non-poor population are required not to make a difference to the extent of measured poverty, then the combination of such a 'population focus axiom' with the replication invariance axiom, in the presence of a number of other canonical fixed and variable population properties such as monotonicity, transfer, maximality, and poverty growth, can be shown to lead to rather elementary impossibility results. Variable populations can thus be a source of difficulty for both the ethical acceptability and logical coherence of poverty measures. The origin of the difficulty can be traced to the (implicitly) inconsistent stance displayed by prevailing measurement approaches to a 'focus axiom' or 'constituency principle', an issue which deserves some discussion. In particular, the second part of this paper will deal with the tendency, which is widely manifest in the poverty measurement literature, to defer to an 'income focus axiom', while denying the (similar) demands of a 'population focus axiom'. the conflicting claims of replication invariance and population focus are more proximately reflected in the conflicting claims of a headcount ratio and an aggregate headcount as the appropriate indicator of the prevalence of poverty. The second part of the present essay will also deal with this issue of 'fractions *versus* whole numbers', and will consider the possible merits of a 'compromise candidate' which combines the headcount ratio and the aggregate headcount in a 'mixed' indicator of the incidence of poverty—an indicator which, arguably, mitigates the problems associated with each of the 'uncontaminated' indicators alluded to earlier.

The third part of the essay will focus on variable populations and inequality measurement. Two very useful properties of inequality measurement are replication invariance (which underlies the construction of the Lorenz curve and is, indeed, at the basis of partial comparisons of distributions such as those facilitated by stochastic dominance criteria) and the normalization axiom, which views all distributions in which a single person appropriates the entire income as reflecting the same (and maximal) extent of inequality. The latter property makes it particularly easy to express the inequality value for an *n*-person distribution in terms of the equivalent share of the poorer of two persons in a classical two-person cake-sharing problem. Unfortunately, it can be shown in a variable population context, that under certain well-defined conditions replication invariance and normalization are mutually incompatible. Some possible ways out of the difficulty (such as via a dilution of the transfer axiom) will be explored.

The paper ends with a summary and conclusions.

2 Preliminaries: concepts and definitions

2.1 Notation

What follows is a presentation of some formal elements of terms and concepts that are of relevance for the measurement of poverty and inequality.

$\mathcal{N}$ will stand for the set of positive integers, $\mathcal{R}$ for the set of real numbers, and $\mathcal{S}$ for the set of positive real numbers. For every $n \in \mathcal{N}$, $\mathbf{X}_n$ will stand for the set of non-decreasingly ordered non-negative n- vectors $\mathbf{x} = (x_1,...,x_i,...,x_n)$, where the typical element x_i of $\mathbf{x}$ stands for the income of person i in a community of n persons. The set of all conceivable income distributions is then given by $\mathbf{X} \equiv \cup_{n\in\mathcal{N}} \mathbf{X}_n$. For every $\mathbf{x} \in \mathbf{X}$, $N(\mathbf{x})$ will designate the set of individuals whose incomes are represented in the vector $\mathbf{x}$, $n(\mathbf{x})$ for the dimensionality of the vector $\mathbf{x}$, and $\mu(\mathbf{x})$ for the mean of the incomes in the vector $\mathbf{x}$. For future reference, we define three distinguished subsets of $\mathbf{X}$: the collection. $\mathbf{X}^*$ of zero vectors, the collection $\hat{\mathbf{X}}$ of '*extremal* distributions' in which all but the richest individual receive zero income while the richest person appropriates the entire income of the society, and the collection $\tilde{\mathbf{X}}$ of equally distributed income vectors:

$\mathbf{X}^* \equiv \{\mathbf{x} \in \mathbf{X} | x_i = 0 \forall i \in N(\mathbf{x})\}$, $\hat{\mathbf{X}} \equiv \{\mathbf{x} \in \mathbf{X} | x_i = 0 \forall i \neq n(\mathbf{x}) \;\&\; x_{n(\mathbf{x})} > 0\}$, and $\tilde{\mathbf{X}} \equiv \{\mathbf{x} \in \mathbf{X} | x_i = \mu(\mathbf{x}) \forall i \in N(\mathbf{x})\}$. *The poverty line*, which is a level of income such that any person with income less than this level will be certified to be *poor*, is designated by z. For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$, $Q(\mathbf{x}; z)$ will stand for the set of poor individuals whose incomes are represented in the income vector $\mathbf{x}$; $q(\mathbf{x}; z)$ for the cardinality of $Q(\mathbf{x}; z)$; $\mathbf{x}_z^P$ for the vector of poor incomes in $\mathbf{x}$; $R(\mathbf{x}; z)$ for the set of non-poor individuals whose incomes are represented in $\mathbf{x}$; $r(\mathbf{x}; z)$ for the cardinality of $R(\mathbf{x}; z)$; and $\mathbf{x}_z^R$ for the vector of non-poor incomes in $\mathbf{x}$.

A *poverty measure* is a mapping $P: \mathbf{X} \times \mathcal{S} \to \mathcal{R}$ such that, for every $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$, $P(\mathbf{x}; z)$ specifies a real number which is supposed to reflect the extent of poverty associated with the regime $(\mathbf{x}; z)$.

An *inequality measure* is a mapping $I: \mathbf{X} \to \mathcal{R}$ such that, for every $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ specifies a real number which is supposed to reflect the extent of inequality associated with the income vector $\mathbf{x}$.

2.2 Axioms for the measurement of poverty

Stated in what follows are a set of fixed-population axioms for poverty measures which have gained a fair amount of consensus in the literature.

Income Focus (*Axiom IF*). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $n(\mathbf{x}) = n(\mathbf{y})$ and $\mathbf{x}_z^P = \mathbf{y}_z^P$, then $P(\mathbf{x}; z) = P(\mathbf{y}; z)$.

Anonymity (*Axiom A*). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y} = \mathbf{\Pi x}$ where $\mathbf{\Pi}$ is some appropriately dimensioned permutation matrix, then $P(\mathbf{x}; z) = P(\mathbf{y}; z)$.

Monotonicity (*Axiom M*; see Hassoun and Subramanian 2011). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $n(\mathbf{x}) = n(\mathbf{y})$, and $x_i = y_i \forall i \in N(\mathbf{x}) \setminus \{j\}$ for some j satisfying $j \in Q(\mathbf{y}; z)$ & $x_j > y_j$, then $P(\mathbf{x}; z) < P(\mathbf{y}; z)$.

Transfer (*Axiom T*; see Hassoun and Subramanian 2011). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $n(\mathbf{x}) = n(\mathbf{y})$, and $x_i = y_i \forall i \in N(\mathbf{x}) \setminus \{j, k\}$ for some j, k satisfying $j \in Q(\mathbf{y}; z)$, $k \in Q(\mathbf{x}; z) \setminus \{j\}$, $x_j = y_j + \delta, x_k = y_k - \delta$, and $0 < \delta \leq (y_k - y_j)/2$, then $P(\mathbf{x}; z) < P(\mathbf{y}; z)$.

Income focus requires measured poverty to be insensitive, other things equal, to increases in non-poor incomes. Anonymity requires the poverty measure to be invariant with respect to permutations of incomes across individuals, so that personal identities do not matter, and this serves as a justification, in cross-section and time-series comparisons of distributions, for seeing one distribution as being derived from another through a population increment or decrement. Monotonicity demands that, other things equal, an increase in a poor person's income should reduce poverty. Transfer (as stated in this paper) is a weak endorsement of equality which requires that a rank-preserving progressive transfer of income from a non-poor person to a poor person, which continues to keep the non-poor person non-poor, should reduce poverty: this is weaker than the weak downward transfer axiom of Donaldson and Weymark (1986).

Following are some variable-population axioms for poverty measurement.

Replication Invariance (*Axiom RI*). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y}$ is a *k*-replication of $\mathbf{x}$, where *k* is any positive integer, that is, if $\mathbf{y} = (\mathbf{x},\mathbf{x},...,\mathbf{x})$ and $n(\mathbf{y}) = kn(\mathbf{x})$, then $P(\mathbf{x}; z) = P(\mathbf{y}; z)$.

Replication Scaling (*Axiom RS*; see Subramanian 2002b). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y}$ is a *k*-replication of $\mathbf{x}$, where *k* is any positive integer, that is, if $\mathbf{y} = (\mathbf{x},\mathbf{x},...,\mathbf{x})$ and $n(\mathbf{y}) = kn(\mathbf{x})$, then $P(\mathbf{y}; z) = kP(\mathbf{x}; z)$.

Weak Poverty Growth (*Axiom WPG*; see Subramanian 2002b). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{x}_z^R = \mathbf{y}_z^R$, $r(\mathbf{x}; z) \geq 1$, $\mathbf{x}_z^P = (x, x,..., x)$ for any $x \geq 0$, $\mathbf{y}_z^P = (x, x,..., x)$, and $q(\mathbf{y}; z) = q(\mathbf{x}; z) + 1$, then $P(\mathbf{x}; z) < P(\mathbf{y}; z)$.

Non-Poverty Growth (*Axiom NPG*; see Kundu and Smith 1983). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y} = (\mathbf{x}, x)$ for any $x \geq z$, then $P(\mathbf{x}; z) > P(\mathbf{y}; z)$.

Weak Population Focus (*Axiom WPF*). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y} = (\mathbf{x}, x)$ for any $x \geq z$, then $P(\mathbf{x}; z) \leq P(\mathbf{y}; z)$.

Population Focus (*Axiom PF*; see Hassoun and Subramanian 2011). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{y} = (\mathbf{x}, x)$ for any $x \geq z$, then $P(\mathbf{x}; z) = P(\mathbf{y}; z)$.

Comprehensive Focus (*Axiom CF*; see Subramanian 2011b). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{x}_z^P = \mathbf{y}_z^P$, then $P(\mathbf{x}; z) = P(\mathbf{y}; z)$.

Maximality (*Axiom MX*; see Subramanian 2011b). For all $\mathbf{x},\mathbf{y} \in \mathbf{X}$ and $z \in \mathcal{S}$, if $\mathbf{x} \in \mathbf{X}^*$ and $\mathbf{y} \notin \mathbf{X}^*$, then $P(\mathbf{x}; z) \geq P(\mathbf{y}; z)$.

Replication invariance is widely perceived to be a very undemanding and reasonable property, which prescribes that measured poverty should depend only on the relative, not the absolute, frequency of incomes in a distribution: it is at the basis of Lorenz and stochastic dominance comparisons of income distributions, and constitutes a virtually universally accepted property of poverty measures, whereby poverty is measured in *per caput* terms. Replication scaling, by contrast, calls for measured poverty to register a *k*-fold increase whenever an income distribution undergoes a *k*-fold replication. Weak poverty growth is a weakened version of a property called 'poverty growth' introduced by Kundu and Smith (1983): the latter condition requires that poverty should increase whenever there is an addition to the poor population, while the former requires that if all the poor in a population that has at least one non-poor person should have the same income, then an addition of a poor person with the same income as the rest of the poor should cause poverty to rise. The non-poverty growth axiom, due to Kundu and Smith (1983), requires that the addition of a non-poor person to the population should cause poverty to decline: implicit in this requirement seems to be an acceptance of the view that the prevalence of poverty is appropriately captured by the *proportion* of a population in poverty. The weak population focus axiom, however, is diametrically opposed in spirit to the non-poverty growth axiom: it reflects the requirement of what Hassoun (2010) calls the 'no mere addition' property, whereby poverty ought not to be seen to decline with the addition of a non-poor person to the population. The population focus axiom is a strengthened version of Hassoun's no mere addition axiom: it reflects what Paxton (2003) calls the 'poverty non-invariance' property, whereby poverty remains unchanged by the addition of a non-poor person to the population. Comprehensive focus—also called 'strong focus' in Subramanian (2002b)—subsumes both the income focus and the population focus axioms, by requiring that poverty ought to remain unchanged following on an increase in either the income of a non-poor person or the size of the non-poor population. Maximality is the requirement that poverty is never greater than when every person in a community has zero income: this is compatible, for instance, with a zero-one normalization of the poverty measure, with the upper-bound of unity reserved for the situation in which every person has zero income—such as would be the case if, following the normalization procedure resorted to by Pattanaik and Sengupta (1995), one were to identify the poverty measure with the proportion of the population in poverty when every person has zero income.

2.3 Some well-known measures of poverty

The Headcount Ratio H. For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$:

$$H(\mathbf{x}; z) \equiv q(\mathbf{x}; z) / n(\mathbf{x}).$$

The headcount ratio is just the proportion of the population in poverty.

The Income-Gap ratio I (see Sen 1976). For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$:

$$I(\mathbf{x}; z) \equiv 1 - \mu^P(\mathbf{x}; z) / z,$$

where $\mu^P(\mathbf{x}; z)$ is the average of poor incomes in the vector $\mathbf{x}$. The income-gap ratio is just the proportionate shortfall of the average income of the poor from the poverty line, or the proportionate poverty gap per poor person.

The Per Capita Income-Gap Ratio R (see Sen 1976). For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$:

$$R(\mathbf{x}; z) \equiv q(\mathbf{x}; z)(z - \mu^P(\mathbf{x}; z)) / n(\mathbf{x})z = \left(\frac{q(\mathbf{x}; z)}{n(\mathbf{x})}\right)\left(1 - \mu^P(\mathbf{x}; z) / z\right) = H(\mathbf{x}; z)I(\mathbf{x}; z).$$

The per capita income-gap ratio is the proportionate poverty gap per person in the general population, and is given by the product of the headcount and the income-gap ratios.

The Sen Index of Poverty S (see Sen 1976). For all $\mathbf{x} \in \mathbf{X}$ such that $\mathbf{x}$ is a non-decreasingly ordered vector of incomes, and $z \in \mathcal{S}$:

$$S(\mathbf{x}; z) \equiv [2 / (q(\mathbf{x}; z) + 1)z] \Sigma_{i \in Q(\mathbf{x}; z)} (z - x_i)(q(\mathbf{x}; z) + 1 - i).$$

For indefinitely large values of $q(\mathbf{x}; z)$, Sen's index can be approximated by the expression

$$S(\mathbf{x};z) = H(\mathbf{x};z)[I(\mathbf{x};z) + (1 - I(\mathbf{x};z))G^P(\mathbf{x}_z^P)],$$

where $G^P(\mathbf{x}_z^P)$ is the Gini coefficient of inequality in the distribution of poor incomes in the vector **x**: the Sen measure, therefore, can be written as a composite function of the incidence of poverty (as captured by the headcount ratio), the depth of poverty (as captured by the income-gap ratio), and the severity of poverty (as captured by the interpersonal inequality in the distribution of poor incomes).

The Foster-Greer-Thorbecke P_α Family of Measures (see Foster, Greer and Thorbecke 1984). For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$:

$$P_\alpha(\mathbf{x};z) \equiv \left(\frac{1}{n(\mathbf{x})}\right)\Sigma_{i \in Q(\mathbf{x};z)}\left(\frac{z - x_i}{z}\right)^\alpha, \alpha \geq 0.$$

Certain distinguished members of the P_α family are the following:

For all $\mathbf{x} \in \mathbf{X}$ and $z \in \mathcal{S}$:

$$P_0(\mathbf{x};z) = H(\mathbf{x};z);$$

$$P_1(\mathbf{x};z) = R(\mathbf{x};z);$$

$$P_2(\mathbf{x};z) = H(\mathbf{x};z)[I^2(\mathbf{x};z) + (1 - I(\mathbf{x};z))^2 C^P(\mathbf{x};z)],$$

where $C^P(\mathbf{x};z)$ is the squared coefficient of variation in the distribution of poor incomes; and in the limit, as α becomes indefinitely large, $P_\alpha(\mathbf{x};z)$ mimics a Rawlsian 'maximin' criterion, whereby the income distributions are ranked solely by the income share of the poorest individual.

As is well-known—see Sen (1976) and Foster et al. (1984)—the headcount ratio (that is to say P_0 or H) violates the monotonicity and transfer axioms, the income-gap ratio (I) and the per capita income-gap ratio (that is to say P_1 or R) satisfy monotonicity while violating transfer, and the Sen Index (S) and P_2 satisfy both monotonicity and transfer. Indeed, the measure P_α satisfies monotonicity for all $\alpha > 0$ and transfer for all $\alpha > 1$. The failure of the headcount ratio to satisfy monotonicity, and the failure of the income-gap ratio and its per capita version to satisfy transfer, were a substantial part of the motivation underlying Sen's (1976) effort to identify a more complete measure that was capable of satisfying these properties: the Sen Index, and members of the P_α family for values of α exceeding unity, are examples of such relatively sophisticated indices of poverty.

2.4 Axioms for the Measurement of Inequality

Some of the axioms for inequality measures are direct counterparts of corresponding axioms for poverty measurement: while the same nomenclature will be adopted for both sets of axioms, the inequality-related axioms will be differentiated from the poverty-related ones by means of a starred designation (so that, for instance, Axiom A* will stand for the anonymity axiom as applied to inequality measures, while Axiom A will stand for the anonymity axiom as applied to poverty measures). First, we present some standard fixed-population axioms for inequality measures.

Anonymity (*Axiom A**). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{y} = \mathbf{\Pi x}$ where $\mathbf{\Pi}$ is some appropriately dimensioned permutation matrix, then $I(\mathbf{x}) = I(\mathbf{y})$.

Transfer (*Axiom T**). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $n(\mathbf{x}) = n(\mathbf{y})$ and $x_i = y_i \forall i \in N(\mathbf{x}) \setminus \{j,k\}$ for some j,k satisfying $x_j = y_j + \delta, x_k = y_k - \delta$, and $0 < \delta \leq (y_k - y_j)/2$, then $I(\mathbf{x}) <$ $I(\mathbf{y})$.

Weak Transfer (*Axiom WT**). Axiom WT* is derived from Axiom T* by replacing the consequent $I(\mathbf{x}) < I(\mathbf{y})$ in the statement of Axiom T* by the weak inequality $I(\mathbf{x}) \leq$ $I(\mathbf{y})$.

Scale Invariance (*Axiom SI*). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{y} = \lambda \mathbf{x}$ where λ is any positive scalar, then $I(\mathbf{x}) = I(\lambda \mathbf{x})$.

Anonymity, in inequality measurement as in poverty measurement, requires the measure to be invariant with respect to personal identities. Transfer requires the inequality measure to register a decline in value whenever a rank-preserving progressive transfer of income between two persons occurs. Weak Transfer is a less demanding requirement, by which inequality should merely not increase following on a progressive rank-preserving transfer of income between two individuals. It is widely held that that fulfilment of the Transfer Axiom is a necessary condition for any inequality measure to qualify *as* an inequality measure; though this view is sometimes disputed, as in the work of Chateauneuf and Moyes (2006). Out of deference to the general view that prevails in this matter, one could call an inequality measure $I : \mathbf{X} \to \mathcal{R}$ *a proper measure of inequality* if and only if for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ satisfies Axiom T*. We could call an inequality measure $I : \mathbf{X} \to \mathcal{R}$ *a threshold measure of inequality* if and only if for all $\mathbf{x} \in \mathbf{X}$, $I(\mathbf{x})$ satisfies Axiom WT* but not Axiom T*. Finally, Scale Invariance requires an inequality measure to be seen in the light of a purely *relative* measure, namely that any uniform scaling up or down of an income vector should leave the extent of measured inequality unchanged.

Next, we present a few variable population inequality measures.

Replication Invariance (*Axiom RI**). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{y}$ is a *k*-replication of $\mathbf{x}$, where *k* is any positive integer, that is, if $\mathbf{y} = (\mathbf{x}, \mathbf{x}, ..., \mathbf{x})$ and $n(\mathbf{y}) = kn(\mathbf{x})$, then $I(\mathbf{x}) = I(\mathbf{y})$.

Upper Pole Monotonicity (*Axiom UPM*; see Subramanian 2010, 2011a). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \in \hat{\mathbf{X}}$ and $\mathbf{y} = (\mathbf{x}, x)$, where x is the income of the richest individual in the income vector **x**, then $I(\mathbf{y}) < I(\mathbf{x})$.

Lower-Bound Normalization (*Axiom LBN*). For all $\mathbf{x} \in \tilde{\mathbf{X}}$, $I(\mathbf{x}) = 0$.

Weak Upper-BoundNormalization (Axiom WUBN; see Subramanian 2010, 2011a). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \in \hat{\mathbf{X}}$ and $\mathbf{y} = (\mathbf{x}, 0)$, then $I(\mathbf{y}) \le I(\mathbf{x})$.

Upper-Bound Normalization (*Axiom UBN*; see Subramanian (2010, 2011a). For all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, if $\mathbf{x} \in \hat{\mathbf{X}}$ and $\mathbf{y} = (\mathbf{x}, 0)$, then $I(\mathbf{y}) = I(\mathbf{x})$.

Replication invariance requires the inequality measure to depend only on the relative, not the absolute, frequency of incomes in a distribution. ‘Upper pole monotonicity’ and ‘upper-bound normalization’ are properties introduced by Subramanian (2010, 2011a), and deal with what ought to be seen to be happening to inequality in an ‘extremal’ distribution (one in which all but the richest individual have no income at all) due to the addition of a person at either end of the distribution. Axiom UPM advances the reasonable requirement that inequality should be seen to be diluted when a person, with the same income as that of the richest individual in an extremal distribution, joins the population. Asymmetrically, however, the upper-bound normalization axiom requires measured inequality to be invariant to the addition of a person with zero income to an extremal distribution. Axiom UBN is analogous to its lower-bound counterpart: lower-bound normalization requires that the extent of inequality should be assessed at zero when there is a perfectly equal division of income in a society, and this same value (of zero) is reserved for *all* distributions—irrespective of their dimensionality—when income is perfectly equally divided amongst the population. In a similar spirit, Axiom UBN requires that no matter what the dimensionality of an income vector is, as long as inequality is as bad as it possibly could be (given the size of the population), that is, as long as a single person appropriates the entire income of a society, the addition to the population of another person with zero income ought to make no difference to the extent of measured poverty. The notion of normalization with respect to the limits that can be achieved in relation to the constraints describing any given situation is well captured in the following apparently flippant passage from Carroll’s *Through the Looking-Glass* (quoted also in Subramanian 2010):

> ‘I like the Walrus best’, said Alice: ‘because he was a *little* sorry for the poor oysters.’
>
> ‘He ate more than the Carpenter, though’, said Tweedledee. ‘You see he held his handkerchief in front, so that the Carpenter couldn’t count how many he took: contrariwise.’
>
> ‘That was mean!’ Alice said indignantly. ‘Then I like the Carpenter best—if he didn’t eat so many as the Walrus.’

'But he ate as many as he could get', said Tweedledum.

This was a puzzler.

2.5 Some well-known real-valued measures of inequality

Following are the expressions for a set of inequality measures which are widely known in the literature (and which will therefore not be discussed here). All these measures satisfy the fixed-population properties of Anonymity, Transfer and Scale Invariance, and the variable population properties of lower-bound normalization, replication invariance and upper pole monotonicity (see Subramanian 2011a).

The Squared Coefficient of Variation (C^2). For all $\mathbf{x} \in \mathbf{X}$:

$$C^2(\mathbf{x}) = (1/n(\mathbf{x})\mu^2(\mathbf{x}))\sum_{i\in N(\mathbf{x})} x_i^2 - 1$$

Theil's Inequality Index (T). For all $\mathbf{x} \in \mathbf{X}$:

$$T(\mathbf{x}) = (1/n(\mathbf{x}))\sum_{i\in N(\mathbf{x})}\left(\frac{x_i}{\mu(\mathbf{x})}\right)\log\left(\frac{x_i}{\mu(\mathbf{x})}\right).$$

The Gini Coefficient of Inequality (G). For all $\mathbf{x} \in \mathbf{X}$:

$$G(\mathbf{x}) = \frac{n(\mathbf{x})+1}{n(\mathbf{x})} - \left(\frac{2}{n^2(\mathbf{x})\mu(\mathbf{x})}\right)\sum_{i\in N(\mathbf{x})}(n(\mathbf{x})+1-i)x_i\,,$$

where individual incomes have been arranged in non-decreasing order, viz. $x_i \le x_{i+1}, i = 1,\dots,n(\mathrm{x})-1$.

The Atkinson Family of Ethical Inequality Indices (A_λ). For all $\mathbf{x} \in \mathbf{X}$:

$$A_\lambda(\mathbf{x}) = 1 - \left[\frac{1}{n(\mathbf{x})\mu(\mathbf{x})}\sum_{i\in N(\mathbf{x})} x_i^\lambda\right]^{\frac{1}{\lambda}}, \lambda \in (0,1).$$

For future reference (see Subramanian 2011a) we also provide the expressions for the normalized versions of the above inequality measures, obtained by dividing each of the measures by the maximum value it can attain (which happens when the distribution is an extremal one); these normalized versions are distinguished by supplying each of the respective measures with a star superscript, so that:

(2.5) $C^2* = \left(\frac{1}{n-1}\right)C^2$,

(2.6) $T* = (1/\log n)T$,

(2.7) $G* = \left(\frac{n}{n-1}\right)G$, and

(2.8) $A_\lambda * = [1/(1-n^{\frac{\lambda-1}{\lambda}})]A_\lambda$.

(Notice that, since there is no ambiguity, we have taken the liberty of writing C^2 for $C^2(\mathbf{x})$, n for $n(\mathbf{x})$, and so on.)

3 Parfit's 'Theory X' and poverty measurement: some parallels

A major concern in Parfit's (1984) book *Reasons and Persons* is with what he calls the 'awesome' question of 'how many people should there ever be?'. This leads him to a consideration of how to assess the well-being of populations of alternative sizes: some satisfactory theory of beneficence is required to address the question of how many people there should ever be, and he calls such a theory of population ethics, assuming it exists and can be discovered, 'Theory X'. Theory X is a theory of the 'good', as captured in what Parfit (1984: 381) refers to as '… the level of happiness, or … the quality of life, or … the share per person of resources. We should assume that, in my examples, these three correlate, rising and falling together.' Parfit's quest for Theory X is informed by the notion that a proper reckoning of well-being should combine information on the following ingredients: the quantity of well-being, the quality of well-being, and the extent of inequality, if any, in the inter-personal distribution of well-being.

It is striking that Sen's (1976) seminal quest for a satisfactory measure of income poverty, which could be seen as a theory of the 'bad', was informed by precisely the considerations that motivated Parfit's Theory X. Recall from the preceding section that, for 'large' numbers of the poor, Sen's poverty index is given by: $S = HI + H(1-I)G^P$. Viewing poverty as an aspect of 'ill-being', it seems reasonable to interpret HI as signifying the quantity of ill-being, I as signifying the quality of ill-being, and G^P as signifying inequality in the inter-personal distribution of ill-being. In essential respects, it can be claimed, Sen's quest for a measure of the 'bad' is reflected in Parfit's quest for a measure of the 'good'. It is interesting to note that Parfit, at the end of his book, concedes his inability to come up with a satisfactory version of Theory X: '… though I failed to find such a theory, I believe that, if they tried, others could succeed' (Parfit 1984: 443). The present author (Subramanian 2006), on which this section is heavily dependent), has demonstrated that it is not just the motivation underlying the Parfit and Sen enterprises that share commonalities, but also their respective outcomes. This is explicated, in what follows, with the help of a number of elementary examples. In all these examples (unless otherwise stated), we shall, for specificity, take it that $z = 100$ and $n = 1$ million.

Consider first the ordered income *n*-vectors $\mathbf{x}^1 = (99,....,99)$ and $\mathbf{y}^1 = (0,...,0)$. One would be normally disposed to imagine that $\mathbf{x}^1$ is, from a poverty point of view, and in terms of both the quantity and quality of deprivation, a superior distribution to $\mathbf{y}^1$. Yet, this judgment is denied by the headcount ratio of poverty, which takes account of neither the quantity nor quality of poverty, concerned, as it is, solely with the proportion of the population in poverty: $H(\mathbf{x}^1; z) = H(\mathbf{y}^1; z) = 1$.

Next, consider the ordered *n*-vectors $\mathbf{x}^2 = (0,99,....,99)$ and $\mathbf{y}^1 = (0,...,0)$. Again, our normal disposition would be to see $\mathbf{y}^1$ as being poverty-wise worse than $\mathbf{x}^2$ from both a quantity and quality perspective on poverty; but again, this judgment would be denied by the poverty measure $P_{\alpha\to\infty}$, since—in terms of the maximin criterion which focuses only on the income-share of the poorest individual - $P_{\alpha\to\infty}(\mathbf{x}^2; z) = P_{\alpha\to\infty}(\mathbf{y}^1; z)$.

Now consider the pair of ordered income *n*-vectors $\mathbf{x}^3 = (0,100,....,100)$ and $\mathbf{y}^1 = (0,...,0)$. In $\mathbf{x}^3$ one person out of a million is subjected to extreme deprivation, while in $\mathbf{y}^1$ every single one of one million persons is subjected to extreme deprivation; yet, poverty as measured by the income-gap ratio I will certify that the two distributions are poverty-wise indistinguishable, for $I(\mathbf{x}^3; z) = I(\mathbf{y}^1; z) = 1$. If this example militates against one's moral intuition in the matter, the following example does even more violence to one's sense of the rightness of things. If $\mathbf{y}^2 = (0.01,...,0.01)$, then measuring poverty by the income-gap ratio would compel us to judge that there is more poverty in the distribution $\mathbf{x}^3$ than in the distribution $\mathbf{y}^2$, since $I(\mathbf{x}^3; z)(=1) > I(\mathbf{y}^2; z)(= 0.9999)$. The trouble arises from the fact that the measure I is concerned solely with a 'quality' view of deprivation: it reflects a shortcoming which Parfit associates with what he calls the 'Average Principle', a shortcoming that is well-illustrated by Parfit's (1984: 406) 'Two Hells' Example (which, with suitable contextual adaptation, is reflected in the examples of the income vectors $\mathbf{x}^3$ and $\mathbf{y}^2$):

> *The Two Hells*: In *Hell One*, the last generation consists of ten innocent people, who each suffer great agony for fifty years. The lives of these people are much worse than nothing. They would all kill themselves if they could. In *Hell Two*, the last generation consists not of ten but of ten million innocent people, who each suffer agony just as great for fifty years minus a day.

It is not only average utilitarianism but also total utilitarianism which falls foul of Parfit's requirement of a satisfactory theory of well-being. The difficulty with what he calls the 'total principle' is illustrated by the following example. Consider the income *n*-vectors $\mathbf{x}^3 = (0,100,....,100)$ and $\mathbf{y}^3 = (99.9999,...,99.9999)$. One would imagine that the very slight sacrifice of 0.0001 unit of income which each of 999,999 people have to make in order to redeem the extreme deprivation of the poorest person in $\mathbf{x}^3$ would be well worth the transition from $\mathbf{x}^3$ to $\mathbf{y}^3$; yet, in terms of a view of poverty which is concerned only with its total quantity, as measured by the product of the headcount and income-gap ratios—which is the per capita income-gap ratio R as also the Sen Index of poverty S (because there is no inequality in the distribution of poor incomes in either $\mathbf{x}^3$ or $\mathbf{y}^3$)—we would be obliged to declare that $R(\mathbf{x}^3; z)[= S(\mathbf{x}^3; z1] = R(\mathbf{y}^3; z)[= S(\mathbf{y}^3; z)](= 10^{-6})$. This result is a version of what Parfit calls the Repugnant Conclusion yielded by the exclusive concern of classical utilitarianism with the 'Total Principle'. Restated in a poverty context, the Repugnant Conclusion would read something like this: 'As long as there is invariance in the total quantity of deprivation that obtains, there is really no moral distinction to be drawn between a situation in which a single person suffers the most extreme deprivation and one in which a sufficiently large number of individuals experience very mild deprivation.'

The repugnant conclusion, it turns out, is a feature of the entire P_α family of poverty indices, for finite integral values of α exceeding unity. To see this, consider a situation in which $n = 10^\alpha$ (where α is a finite integer greater than one), and we have the income n-vectors $\mathbf{x}^3 = (0,100,....,100)$ and $\mathbf{y}^4 = (90,...,90)$. Again, and for the same reasons that were advanced in favour of $\mathbf{y}^3$ over $\mathbf{x}^3$, one imagines one would be inclined to favour $\mathbf{y}^4$ over $\mathbf{x}^3$ from a poverty point of view. However, it can be verified that for all finite integral values of α exceeding one, $P_\alpha(\mathbf{x}^3; z) = P_\alpha(\mathbf{y}^4; z)(= 1/n)$.

The very elementary examples employed above suggest that none of the poverty indices considered in this paper—the headcount ratio, the income-gap ratio, the per capita income gap-ratio, the Sen Index of poverty, or the entire Foster-Greer-Thorbecke P_α family of indices—escapes conflicting with one's reasonable moral intuition, in specific cases, on the poverty ranking of distributions. The ingredients of Parfit's Theory X have, by and large, been the ingredients of standard measures of poverty advanced in the literature. Just as Parfit points to the inadequacies of the total and average principles, and the possibility of a repugnant conclusion, in the context of variable population well-being comparisons, so one encounters analogous and problematic versions of the total and average principles, and a version of the repugnant conclusion, in the context of poverty comparisons across poor populations of variable size (even when the aggregate population is of fixed dimension). Parfit's verdict of a failure in his quest for Theory X would thus also appear to hold for the economist's quest for a satisfactory real-valued measure of poverty. A different set of problems, again with close links to difficulties which have been noted in the literature on population ethics, arises when we resort to poverty comparisons across populations of variable aggregate size, an issue to which we now turn.

4 Variable population poverty comparisons

Virtually all extant measures of poverty emphasize a *headcount ratio*, rather than an *aggregate headcount*, view of poverty. This, in turn, is because virtually all extant measures of poverty either explicitly or implicitly endorse the replication invariance axiom and deny the population focus axiom, even as they accept the income focus axiom. To see the relationship between the headcount ratio and replication invariance, and the relationship between the aggregate headcount and population focus, note first that, under any k-fold replication of an income distribution, the headcount ratio will remain unaffected, while the aggregate headcount will register a *k*-fold increase; and, second, with an addition to the nonpoor population, the aggregate headcount will remain unaffected, while the headcount ratio will register a decline. It would appear to be inconsistent to find merit in the income focus axiom and none in the population focus axiom; when this inconsistency is sought to be rectified by requiring poverty indices to also satisfy population focus, then we find—unsurprisingly perhaps, but also disquietingly—that Population Focus in conjunction with other axioms which traditionally emphasize a headcount ratio view of poverty leads to incoherence and impossibility. This section—which relies heavily on Subramanian (2002b, 2011b), Hassoun (2010) and Hassoun and Subramanian (2011)—presents a small set of very elementary impossibility theorems which point to the difficulties inherent in variable population poverty comparisons.

Proposition 1. There exists no anonymous poverty measure $P:\mathbf{X}\times\mathcal{S}\rightarrow\mathcal{R}$ which satisfies replication invariance (Axiom RI), weak poverty growth (Axiom WPG), and weak population-focus (Axiom WPF).

Proof. Let the poverty line be z, and let x and y be two levels of income such that $x \le z < y$. Consider the income distributions $\mathbf{a}=(x,y), \mathbf{b}=(x,y,y)$ and $\mathbf{c}=(x,x,y,y)$. By Axiom WPG, $P(\mathbf{b};z)<P(\mathbf{c};z)$, and by Axiom RI, $P(\mathbf{c};z)=P(\mathbf{a};z)$, whence $P(\mathbf{b};z)<P(\mathbf{a};z)$ - which, however, is contradicted by $P(\mathbf{a};z)\le P(\mathbf{b};z)$, as dictated by Axiom WPF. ■

Proposition 2. There exists no anonymous poverty measure $P:\mathbf{X}\times\mathcal{S}\rightarrow\mathcal{R}$ which satisfies maximality (Axiom MX), weak poverty growth (Axiom WPG), and weak population focus (Axiom WPF).

Proof. Let the poverty line be z, and let x be a level of income satisfying $x\ge z$. Consider the income distributions $\mathbf{a}=(0,...,0)$, $\mathbf{b}=(\mathbf{a},x)$ and $\mathbf{c}=(\mathbf{b},0)$. We now have: $P(\mathbf{c};z)>P(\mathbf{b};z)$ by Axiom WPG, and $P(\mathbf{b};z)\ge P(\mathbf{a};z)$ by Axiom WPF, whence $P(\mathbf{c};z)>P(\mathbf{a};z)$ which, however, is contradicted by $P(\mathbf{a};z)\ge P(\mathbf{c};z)$, as dictated by Axiom MX. ■

Proposition 3. There exists no anonymous poverty measure $P:\mathbf{X}\times\mathcal{S}\rightarrow\mathcal{R}$ which satisfies monotonicity (Axiom M), replication invariance (RI), and population focus (Axiom PF).

Proof. Let the poverty line be z, and let x and y be two levels of income such that $0 \le x<z<y$. Consider the income distributions $\mathbf{a}=(x,x,...,x,x), \mathbf{b}=(x,x,...,x,y)$ and $\mathbf{c}=(x,x,...,x)$, with $n(\mathbf{a})=n(\mathbf{b})=n(\mathbf{c})+1$. Let $\mathbf{d}=(\mathbf{a},...,\mathbf{a})$ be a fourth income vector such that $n(\mathbf{d})=n(\mathbf{c})n(\mathbf{a})$. It is easy to see that, also, $\mathbf{d}=(\mathbf{c},...,\mathbf{c})$, with $n(\mathbf{d})=n(\mathbf{a})n(\mathbf{c})$. By Axiom RI, one must have $P(\mathbf{a};z)=P(\mathbf{d};\mathrm{z})$ and $P(\mathbf{d};z)=P(\mathbf{c};\mathrm{z})$, whence $P(\mathbf{a};z)=P(\mathbf{c};\mathrm{z})$; this, coupled with $P(\mathbf{a};z)>P(\mathbf{b};\mathrm{z})$ as dictated by Axiom M, leads to

$P(\mathbf{c};z) > P(\mathbf{b};\mathrm{z})$ - which, however, falls foul of what Axiom PF implies, namely $P(\mathbf{c};z) = P(\mathbf{b};\mathrm{z})$. ■

Proposition 4 (***Corollary to Proposition 3***). There exists no anonymous poverty measure $P: \mathbf{X} \times \mathcal{S} \to \mathcal{R}$ which satisfies transfer (Axiom T), replication invariance (RI), and population focus (Axiom PF).

Proof. The proof follows, given Proposition 3, from the fact that Axioms T and PF together imply Axiom M. To see this, imagine a situation in which z is the poverty line, n is a positive integer, Δ is a positive scalar, and $\mathbf{x}, \mathbf{y}, \mathbf{u}$ and $\mathbf{v}$ are four income vectors satisfying $\mathbf{x} = (x_1, ..., x_n)$; $\mathbf{y} = (y_1, ..., y_n)$, with $y_i = x_i \forall i \neq j$ for some $j \in Q(\mathbf{x}; z)$ and $y_j = x_j + \Delta$; $\mathbf{u} = (\mathbf{x}, z + \Delta)$; and $\mathbf{v} = (\mathbf{y}, \mathrm{z})$. By Axiom PF, $P(\mathbf{v}; z) = P(\mathbf{y}; z)$ and by Axiom T, $P(\mathbf{u}; z) > P(\mathbf{v}; z)$, whence $P(\mathbf{u}; z) > P(\mathbf{y}; z)$, which, together with $P(\mathbf{u}; z) = P(\mathbf{x}; z)$ as implied by Axiom PF, leads to $P(\mathbf{x}; z) > P(\mathbf{y}; z)$ - which, precisely, is what is dictated by Axiom M. We have shown that Axioms T and PF in conjunction imply Axiom M; from Proposition 3, we know that there exists no anonymous poverty measure $P: \mathbf{X} \times \mathcal{S} \to \mathcal{R}$ which simultaneously satisfies Axioms M, RI and PF; it follows that there exists no anonymous poverty measure $P: \mathbf{X} \times \mathcal{S} \to \mathcal{R}$ which simultaneously satisfies Axioms T, RI and PF. ■

Propositions 1 and 2 are based on results available in Subramanian (2002b) and Subramanian (2011b) respectively, while Propositions 3 and 4 are available in Hassoun (2010) and Hassoun and Subramanian (2011). The impossibility results stated and proved above are fairly straightforward ones, and require little in the way of complicated reasoning to comprehend. The implications of these results, however, are of some significance for the measurement of poverty. In particular—and as argued in Hassoun (2010) and in Subramanian (2011b)—it would appear that there are at least two possible views one may take of what one calls ‘a measure of poverty’. Under the first view, one measures ‘how poor a society is’; under the second view, one measures ‘how much poverty there is in a society’. The latter view would deem all information relating to the status of the non-poor population as being irrelevant for a measure of poverty, but not so the former. The latter view, that is, would defer to a focus axiom or what, in more general terms, Broome (1996) refers to as a ‘constituency principle’ of population ethics: the principle that, in comparing the ‘goodness’ of alternative states of the world, one takes account only of how good the states are for the *relevant constituency of individuals*, namely those individuals only—such as those that exist in both the states under review—whose preferences and interests can be validly seen to matter for the comparison.

In the context of poverty measurement, it is arguable that the poverty ranking of alternative distributions must depend solely on the interests and preferences of the *poor* constituency of the population. What is important to note is that if such a view is to be defended, it must be defended in its entirety, that is to say, one must defer to what in section 2 has been labelled a ‘comprehensive focus axiom’, one which respects both income focus *and* population focus. Alternatively, one may reject both the income focus and the population focus axioms. An index that satisfies comprehensive focus is any standard measure of poverty which incorporates the headcount ratio, such as the Sen

Index, multiplied by the total population: the headcount ratio in the expression for the Sen Index would then be replaced by the aggregate headcount (call it *A*), and the resulting measure (call it $S' \equiv A[I+(1-I)G^P]$) would defer to both income focus and population focus. An example of a measure which *violates* both income focus and population focus is Anand's (1977) modification of the Sen Index, given, for 'large' numbers of the poor, by the expression $S'' = H[I^* + (1-I^*)G^P]$ where I^* is a modified income-gap ratio which measures the shortfall of the average income of the poor from the poverty line as a proportion of the average income of the entire population rather than of the poverty line ($I^* \equiv 1-\mu^P/\mu$, and μ is the average income of the entire population). Without entering into the substantive merits of a constituency principle, one may still pronounce on a matter of consistency, as such: namely, that it would be consistent to violate both income and population focus, or to respect comprehensive focus, but inconsistent to defer to one of the focus axioms while violating the other. In this sense, the measure S' is a consistent measure (in that it satisfies both income and population focus), just as the measure S'' is also a consistent measure (in that it violates both income and population focus), whereas, unfortunately, most extant measures of poverty are inconsistent, in that they tend to insist on the sanctity of income focus, while apparently seeing no case for population focus. It is this inconsistency which is at the heart of the impossibility results subsumed in Propositions 1-4: replication invariance and maximality are properties of a poverty measure which uphold a 'how poor a society is' view of poverty, while population focus is a property that upholds a 'how much poverty there is in a society' view of poverty. Combining these conflicting views of poverty inevitably leads to incoherence.

An issue that is directly precipitated by the above considerations has to do with the rival claims of the headcount ratio (H) and the aggregate headcount (A) as the appropriate indicator of the prevalence of poverty. This problem has been considered in Subramanian (2005a and 2005b), and in Chakravarty et al. (2006). Perhaps one of the earliest efforts at dealing with the problem from a conceptual perspective is to be found in related work done by Arriaga (1970) on the measurement of urbanization. As pointed out in Subramanian (2005a, 2005b), the headcount ratio violates, and the aggregate headcount satisfies, the constituency principle; on the other hand, the headcount ratio satisfies, and the aggregate headcount violates, what one may call a 'likelihood principle', which is the principle that an assessment of the extent of poverty in a population should carry some indication of the probability of encountering a poor person in that population. Thus, arguably, each of H and A has something to commend it, but each also has something to detract from it. Under the circumstances, it may always be best, in empirical work dealing with the prevalence of poverty, to report on both the headcount ratio *and* the aggregate headcount. This is not a particularly common practice, but two notable exceptions are reflected in the work of Sundaram and Tendulkar (2003), and Reddy and Miniou (2007).

An alternative to providing a disaggregated picture of the headcount ratio and the aggregate headcount is to *combine* the two indices in a composite headcount indicator of poverty. Examples of this approach are available in Arriaga (1970), Chakravarty et al. (2006), and Subramanian (2005a). The last-cited work advances an axiom of 'flexible replication responsiveness' (Axiom FRR), in terms of which a *k*-fold replication of an income distribution induces a $k^\beta-$fold increase in the extent of measured poverty, where β is a parameter in the interval (0,1): the closer β is to zero, the closer the FRR Axiom is to replication invariance; and the closer β is to unity, the closer the FRR Axiom is to replication scaling. If we pitch β at the mid-point (1/2) of the unit interval, then a 'compromise headcount index' which combines the headcount ratio and the aggregate headcount in a 'mixed' measure is given—under some reasonably undemanding axiomatic restrictions—by the quantity $M \equiv A^{\frac{1}{2}}(1+H)$, a measure which has been advanced and discussed in Subramanian (2005a). A possibly useful feature of the measure M is that when two income distributions are indistinguishable in terms of the headcount ratio, M ranks the distributions according to the aggregate headcount; and when two distributions are indistinguishable in terms of the aggregate headcount, M ranks the distributions according to the headcount ratio.

The competing appeals of H and A are, in the end, only a specific manifestation of the more general conceptual difficulties that preside over an appropriate interpretation of what it means to measure 'the extent of poverty' in situations—which are the rule rather than the exception—wherein poverty comparisons have to be effected across populations of variable size. This section has provided a summary of some of these difficulties relating to poverty measurement and population ethics. A similar exercise is undertaken, in the following section, on problems relating to inequality measurement and population ethics.

5 Variable population inequality comparisons

When we deal with variable populations we find that the problem of 'fractions versus whole numbers' encountered in the measurement of poverty carries over also to the measurement of inequality. This is far from surprising: to recall from Section 2, properties such as replication invariance are concerned with population proportions, while properties such as upper pole monotonicity are concerned with absolute population size. The conflict between these two ways of viewing population size is manifested in the following elementary impossibility result, stated and proved in Subramanian (2010) and Subramanian (2011a):

Proposition 5. There exists no anonymous inequality measure $I:\mathbf{X}\to\mathcal{R}$ which satisfies upper pole monotonicity (Axiom UPM), replication invariance (Axiom RI*) and weak upper-bound normalization (Axiom WUBN).

Proof. Let x be any positive scalar, and let **a, b**, **c** and **d** be four income vectors such that $\mathbf{a}=(0,0,...,0,x), \mathbf{b}=(0,0,...,0,0,x)$, $\mathbf{c}=(0,0,...,0,0,x,x)$, and $\mathbf{d}=(0,0,...,0,x,x,...,x)$, with $n(\mathbf{c})=n(\mathbf{b})+1$, $n(\mathbf{b})=n(\mathbf{a})+1$, and $n(\mathbf{d})=n(\mathbf{a})n(\mathbf{c})$. Then, **d** is an $n(\mathbf{c})$-replication of **a** and an $n(\mathbf{a})$-replication of **c**, so that, by Axiom RI, $I(\mathbf{d})=I(\mathbf{a})$, $I(\mathbf{d})=I(\mathbf{c})$, and therefore $I(\mathbf{a})=I(\mathbf{c})$; and $I(\mathbf{c})<I(\mathbf{b})$ by Axiom UPM, whence $I(\mathbf{a})<I(\mathbf{b})$, which, however, is contradicted by $I(\mathbf{a})\geq I(\mathbf{b})$, as dictated by Axiom WUBN. ■

The result above is reflected in the fact that each of the inequality measures C^2, T, G and A_λ presented in Section 2 satisfies Axiom UPM and RI* while violating Axiom WUBN, and each of the inequality measures $C^2*, T*, G*$ and $A_\lambda*$ satisfies Axiom

UBN and UPM while violating Axiom RI*. This suggests an inherent tension between replication invariance and upper-bound normalization, as is, indeed, confirmed by the following result stated and proven in Subramanian (2011a):

Proposition 6. There exists no proper, anonymous and scale-invariant measure of inequality $I : \mathbf{X} \to \mathcal{R}$ which satisfies replication invariance (Axiom RI*) and weak upper-bound normalization (Axiom WUBN).

Proof. Let x be any positive scalar, and **a**, **b**, **c**, **d**, **e** and **f** be six income vectors such that $\mathbf{a} = (0, x)$, $\mathbf{b} = (0,0, x)$, $\mathbf{c} = (0,0,0, x)$, $\mathbf{d} = (0,0,0,2x)$, $\mathbf{e} = (0,0, x, x)$ and $\mathbf{f} = (0,0,0,0, x, x, x, x)$. By Axiom WUBN, $I(\mathbf{a}) \geq I(\mathbf{b}) \geq I(\mathbf{c})$, whence $I(\mathbf{a}) \geq I(\mathbf{c})$; noting that $\mathbf{d} = 2\mathbf{c}$, scale invariance (Axiom SI) requires that $I(\mathbf{c}) = I(\mathbf{d})$ and hence (since $I(\mathbf{a}) \geq I(\mathbf{c})$), $I(\mathbf{a}) \geq I(\mathbf{d})$. Further, $I(\mathbf{d}) > I(\mathbf{e})$ by Axiom T*, whence, given $I(\mathbf{a}) \geq I(\mathbf{d})$, one must also have $I(\mathbf{a}) > I(\mathbf{e})$. Since **f** is a 2-replication of **e** and a 4-replication of **a**, Axiom RI* dictates that $I(\mathbf{f}) = I(\mathbf{e})$ and $I(\mathbf{f}) = I(\mathbf{a})$, whence $I(\mathbf{a}) = I(\mathbf{e})$ which, however, is contradicted by $I(\mathbf{a}) > I(\mathbf{e})$, as deduced earlier. ■

From a wholly pragmatic point of view, inequality measurement without replication invariance is hard to conceive of: one would have to dispense with such devices of comparison as stochastic dominance and Lorenz dominance, which are foundational aspects of inequality measurement as it is 'standardly' practiced, if one were to renounce replication invariance. Upper-bound normalization is also a practically useful property in an inequality index: it permits one to express the extent of inequality in any general *n*-person distribution in terms of the share of the poorer of two individuals in a classic, and easily comprehended, two-person cake-sharing problem—the equivalence between *n*-person inequality measures and two-person shares is dealt with in Subramanian (2002a) and Shorrocks (2005). If, from these pragmatic considerations of manipulability and interpretability, one wished to retain the properties of replication invariance and upper-bound normalization, then one would have to be prepared to sacrifice certain other properties of an inequality measure. Propositions 5 and 6 suggest that one may have to give up the variable population property of upper pole monotonicity and the fixed population property of transfer in this cause. It turns out, as it happens, that there does exist a 'threshold' inequality measure—namely one which satisfies the weak transfer but not the transfer axiom—which fulfils the requirements of replication invariance and upper-bound normalization, while violating upper pole monotonicity. This result, which is discussed in Subramanian (2011a), is reflected in the following proposition:

Proposition 7. There exists a 'threshold' inequality measure $I : \mathbf{X} \to \mathcal{R}$ which satisfies replication invariance (Axiom RI*) and upper-bound normalization (Axiom UBN).

Proof. Consider the inequality measure $D : \mathbf{X} \to \mathcal{R}$ which, for all $\mathbf{x} \in \mathbf{X}$, is given by:

(1) $D(\mathbf{x}) = 1 - \left(\dfrac{1}{n(\mathbf{x})\mu^2(\mathbf{x})} \right) \Sigma_{i \in N(\mathbf{x})} x_i x_{n(\mathbf{x})+1-i}$,

where the incomes in the vector have been arranged in non-descending order.

Note first, in view of (1), that for any extremal distribution $\mathbf{x} \in \hat{\mathbf{X}}$, $D(\mathbf{x}) = 1$, which establishes that D satisfies Axiom UBN. Next, for any ordered n-vector of incomes $\mathbf{x}$, let $\mathbf{y}$ be a k-fold replication of $\mathbf{x}$, where k is any positive integer. Then, given (1), one can see that $D(\mathbf{y}) = 1 - \left(\frac{1}{kn(\mathbf{x})\mu^2(\mathbf{y})}\right) k\Sigma_{i \in N(\mathbf{x})} x_i x_{n(\mathbf{x})+1-i}$ = $1 - \left(\frac{1}{n(\mathbf{x})\mu^2(\mathbf{x})}\right)\Sigma_{i \in N(\mathbf{x})} x_i x_{n(\mathbf{x})+1-i}$ (since, obviously, $\mu(\mathbf{y}) = \mu(\mathbf{x})$) = $D(\mathbf{x})$, as required to establish Axiom RI*. That D is not violative of 'equity-consciousness' is clear from the fact that D resorts to a weighting structure in which the ith poorest person's income is weighted by the (n+1-i)th poorest person's income, which ensures a non-increasing scheme of weights and, therefore, the fulfilment by D of at least weak transfer. More formally, let $\mathbf{x}$ and $\mathbf{y}$ be two ordered n-vectors of income with the same mean μ, and suppose the antecedents of the transfer and the weak transfer axioms, as stated in Section 2, to be satisfied. It can be verified that $D(\mathbf{x})$—$D(\mathbf{y})$ = $(2\delta / n\mu^2)(x_{n+1-j} - x_{n+1-k})$ ≥ 0, since $x_{n+1-j} - x_{n+1-k} \geq 0$ (which follows from the fact that incomes have been arranged in non-decreasing order), which is what is required to establish weak transfer. (However the regular transfer axiom may be violated: if it should turn out that $x_{n+1-j} = x_{n+1-k}$, then one would have $D(\mathbf{x})$ = $D(\mathbf{y})$, a case where weak transfer, but not transfer, is satisfied.) ■

It may be added that the index D, by virtue of being normalized, lends itself to interpretation in terms of the simplest and most familiar representation of inequality one can think of—the share of the poorer person in the division of a cake of fixed size between two individuals. To see what is involved—the reader is also referred, in this connection, to Shorrocks (2005) and Subramanian (2002a, 2010, 2011a)—consider the following. For any n-person ordered income vector $\mathbf{x}$ with mean μ and inequality value D, construct what may be called a 'dichotomously allocated equivalent distribution' (DAED), which is the two person non-decreasingly ordered income vector $\mathbf{x}^* = (x_1^*, x_2^*)$ with the feature that its mean μ^* is the mean μ of $\mathbf{x}$, and its inequality value D^* is the inequality value D of $\mathbf{x}$. $\mu^* = \mu$ and $D^* = D$ entail, respectively (given (1)), that $x_1^* + x_2^* = 2\mu$ and $1 - x_1^* x_2^* / \mu = D$. Solving for x_1^* and x_2^* from this pair of equations yields:

(2) $x_1^* = \mu(1 - \sqrt{D}), x_2^* = \mu(1 + \sqrt{D})$.

If we designate by $\sigma_D[\equiv x_1^* / (x_1^* + x_2^*)]$ the share of the poorer of the two individuals in the DAED $\mathbf{x}^*$, then, in view of (2), one obtains the following expression for σ_D in terms of the inequality index D:

(3) $\sigma_D = (1 - \sqrt{D}) / 2$.

This relationship is of considerable value in interpreting the ‘meaning’ of the inequality measure. Thus, if in some actual situation involving an n-person distribution the extent of inequality as measured by D should be of the order of 0.25, then this is ‘equivalent’—in view of (3)—to a situation in which the poorer of two persons in a two-person distribution of a cake receives 25 per cent of the cake. The utility of this ‘interpretational advantage’ must, of course, be set off against the fact that a measure such as D is not a ‘proper’, but only a ‘threshold’, measure of inequality, and it does not satisfy the property of upper pole monotonicity.

In a general way, Proposition 7 suggests the existence of a tradeoff amongst competing properties of an inequality index. How the tradeoff is resolved must depend on the value system of the practitioner. What Proposition 7 does do is to indicate that a tradeoff cannot be avoided.

6 Summary and conclusions

This paper has been a selective review of certain implications of variable population comparisons for the ethical content and logical coherence of poverty and inequality measurement. In the first instance, we have considered comparisons for aggregate populations of fixed size in which, however, the size of the poor population is allowed to vary. We find certain parallels between Parfit’s quest for a satisfactory theory of beneficence and the economist’s quest for a satisfactory measure of poverty such that, in particular, the categories of both ‘quantity’ and ‘quality’ (of well-being/poverty) can be adequately reflected in the theory/measure of one’s choice.

Next, we consider variable population poverty comparisons, and note the fact that most available indices of poverty insist on the fulfilment of an income focus axiom, the spirit of which, however, is violated by non-observance of an analogous population focus axiom. Population focus, in conjunction with other canonical fixed and variable population axioms (in particular, replication invariance) is found to result in impossibility theorems. This suggests the need for a consistent stance to be displayed toward a constituency principle, one in which both income and population focus are respected, or neither is (or, indeed, the need for a ‘poverty line’ separating the poor from the non-poor is altogether dispensed with, such as would happen with a wholly ‘fuzzy’ approach to poverty conceptualization, in which all individuals are seen to be more or less poor, rather than as either poor or non-poor).

Finally, we note that inequality comparisons across variable populations are also not devoid of complication, and depending upon what particular combination of fixed and variable population axioms we may find relatively attractive, we may be compelled to choose amongst alternative combinations of properties. In general, variable population poverty and inequality comparisons have tended to be somewhat facilely performed through the postulation of a replication invariance axiom, and the present paper has been concerned to argue that the logical and ethical implications of this standard scheme of resolution may be open to question.

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